Estimating the Manifold Dimension

流形维数估计

of Causal Sets

因果集方法

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Abstract

摘要

In a causal set theory of spacetime, on macroscopic scales, physically relevant causal sets must approximate spacetime manifolds. Methods for extracting manifold-like properties from causal sets are an active area of research. This chapter offers a pedagogical treatment of several methods that have been developed and used to estimate the dimension of a manifold that is approximated by a causal set. We also devote some brief comments to the value of dimension estimation to topics of current interest besides estimating the manifold dimension.

在时空的因果集合理论中，宏观尺度下符合物理实际的因果集合必须能够近似时空流形。从因果集合中提取流形性质的方法是当前一个活跃的研究领域。本章对已开发并用于估算因果集合所近似的流形维度的若干方法做教学性介绍。我们还简要讨论了除估算流形维度外，维度估算对当前其他热门研究课题的价值。

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Introduction

引言

Causal set theory ultimately seeks to put forward a complete, self-consistent model of quantum gravity. However, one of its founding principles is that under appropriate macroscopic conditions, spacetime manifolds will emerge from the information encoded in the causal sets. This property is desired for correspondence with general relativity. A basic property of such manifolds is their dimension. Knowing the dimension of a spacetime is key to a proper interpretation of many aspects of the physics of that spacetime. The dimension is also required input for the calculation of many important quantities.

因果集理论最终旨在提出一个完整自洽的量子引力模型。但其奠基原则之一指出，在适当的宏观条件下，时空流形将从因果集编码的信息中涌现出来。为了与广义相对论对应，这一性质是必要的。这类流形的一个基本性质就是其维度。了解时空的维度是正确阐释该时空诸多物理性质的关键，维度也是许多重要物理量计算所需的输入量。

Broadly, within physics and mathematics, there are several kinds of dimension so it is important for us to be clear about what we mean. As alluded to in the previous paragraph, our focus will be (mostly) on the macroscopic limit in which a causal set may possess properties consistent with a spacetime manifold. One way to characterize this manifoldlikeness is to say that the set can be embedded into a manifold with uniform density. The dimension of this macroscopic spacetime manifold into which a causal set can embed is the primary focus of this chapter-we refer to this as the manifold dimension (This term is not universal and there is no standard. Several other terms appear in the literature depending on the context, including spacetime dimension, topological dimension, Minkowski dimension, and even the broader term Hausdorff dimension depending on how it is calculated.).

广义而言，在物理和数学领域存在多种维度，因此明确我们所讨论的维度含义十分重要。正如上一段所提到的，我们的研究 (主要) 聚焦于宏观极限: 因果集在该极限下可拥有与时空流形一致的性质。刻画这种类流形性质的一种方式是，该集合可以均匀密度嵌入到一个流形中。因果集所能嵌入的这个宏观时空流形的维度，是本章的核心研究对象——我们将其称为流形维度 (该术语并非通用，目前也没有标准叫法。文献中根据上下文会使用其他多个术语，包括时空维度、拓扑维度、闵可夫斯基维度，根据计算方式甚至会使用更宽泛的豪斯多夫维度)。

In this chapter, we outline several ways to estimate the manifold dimension of causal sets. The methods we discuss fall into two broad categories: (a) order-theoretic methods that rely on combinatorial properties of causal sets (counting elements and other substructures) to infer the dimension, and (b) spectral dimension methods that infer the dimension from the properties of a random walk. We do not attempt to discuss every possible manifold dimension estimator. Indeed, any combinatorial property of a manifold-like causal set that should depend on its dimension can be used as a dimension estimator. Instead, we focus on those methods that have been most prominently employed in the literature and those that are of current interest.

本章将概要介绍估算因果集流形维度的若干方法。我们讨论的方法大致分为两类:(a) 序理论方法，这类方法依靠因果集的组合性质 (对元素及其他子结构计数) 来推断维度；(b) 谱维度方法，这类方法从随机游走的性质出发推断维度。我们不会逐一讨论所有可能的流形维度估算方法: 实际上，类流形因果集任何依赖于维度的组合性质都可被用作维度估算量。我们转而聚焦文献中最常用、且目前仍受关注的方法。

The remainder of this chapter is organized as follows. In section ”Causal Set Theory”, we give a brief review of causal sets stating several properties and definitions that will be important to our subsequent discussion of dimension estimators. In sections ”Myrheim-Meyer Dimension Estimators” and ”Other Order-Theoretic Dimension Estimators”, we discuss several order-theoretic methods and offer some comments on how they compare. Section ”Spectral Dimension Estimators” treats this class of dimension estimators. In section ”Beyond the Manifold Dimension”, we comment on the application of these techniques to topics other than estimating the manifold dimension. In the final sections, we discuss numerical tests of several dimension estimators and offer some concluding remarks.

本章剩余部分结构安排如下: 在“因果集理论”一节中，我们简要回顾因果集，给出若干对后续讨论维度估算量十分重要的性质与定义；在“迈尔海姆-迈耶维度估算量”和“其他序理论维度估算量”两节中，我们讨论多种序理论方法，并对不同方法的对比做出评述；“谱维度估算量”一节讨论这类维度估算方法；在“流形维度之外”一节中，我们探讨这些技术除估算流形维度外，在其他课题中的应用；在最后几节，我们讨论多个维度估算量的数值检验，并给出结论性评述。

Causal Set Theory

因果集理论

Here we briefly highlight a few particular features of causal sets. A more complete explanation of everything discussed in this section, and many other topics, can be found in the review article by Surya [1].

我们在此简要介绍因果集的若干特殊性质。本节讨论的所有内容以及诸多其他主题的更完整说明，可参见 Surya 撰写的综述文章 [1]。

A causal set C is a set of elements $\{e_i\}$ together with an ordering relation $\prec$ between them. The notation $e_i \prec e_j$ means that e_i precedes e_j . The action of $\prec$ between elements is intended to model the causal ordering of events in spacetime. Therefore, we adopt the following properties: (a) Not all elements are related by the ordering relation so that a causal set is a partially ordered set. (b) The ordering operation is transitive: $e_i \prec e_j$ and $e_j \prec e_k \Rightarrow e_i \prec e_k$, and (c) acyclic: $e_i \prec e_j \Rightarrow e_j \nprec e_i$ for all ordered pairs. The question of reflexivity, whether an element is related to itself, can be chosen either way for convenience.

因果集 C 是由元素集合 $\{e_i\}$ 和元素间的序关系 $\prec$ 共同构成的。记号 $e_i \prec e_j$ 表示 e_i 先于 e_j 。元素间的 $\prec$ 作用是为了建模时空中事件的因果序。因此我们约定如下性质:(a) 并非所有元素都可通过序关系关联，因此因果集是偏序集。(b) 序关系满足传递性: 若 $e_i \prec e_j$ 且 $e_j \prec e_k \Rightarrow e_i \prec e_k$ ，(c) 不存在环: 对所有有序对都满足 $e_i \prec e_j \Rightarrow e_j \nprec e_i$ 。至于自反性 (即元素是否与自身满足序关系)，可根据便利性任意选择。

In addition to the properties just listed, causal set theory adds the feature of spacetime discreteness. This is accomplished by requiring C to be locally finite in the following sense. Consider two related elements $e_i \prec e_k$. Let the order interval $I\left(e_i, e_k\right)$, or interval for short, be the set of elements between them as determined by $\prec$:

除了刚刚列出的性质外，因果集理论还附加了时空离散性的特征。这一点是通过要求 C 满足下述意义的局部有限性实现的: 考虑两个存在序关系的元素 $e_i \prec e_k$ ，将序区间 $I\left(e_i, e_k\right)$ (简称区间) 定义为由 $\prec$ 确定的、位于这两个元素之间的元素集合:

$$I\left(e_i, e_k\right) \equiv \left\{e_j \mid e_i \prec e_j \prec e_k\right\}. \tag{1}$$

For C to be locally finite means that the number of elements in any order interval, called its cardinality $\left|I\left(e_i, e_k\right)\right|$, is finite.

C 是局部有限的，意味着任意序区间内的元素数量 (称为该区间的基数 $\left|I\left(e_i, e_k\right)\right|$) 都是有限的。

Manifold-Like Causal Sets

类流形因果集

As discussed in section "Introduction", our particular focus is on manifold-like causal sets. For such causal sets there are two key correspondences that we highlight here: the number-volume correspondence and the path-geodesic correspondence.

正如在“引言”部分讨论的，我们特别关注类流形因果集。对于这类因果集，我们在此强调两个关键对应关系: 数量-体积对应关系与路径-测地线对应关系。

At its core, causal set theory, in addition to being a discrete model of spacetime, is also a stochastic model of spacetime. Therefore, we should think of the cardinality of an order interval, N , as a fluctuating quantity. Assuming for convenience that a manifold-like causal set has been embedded into a manifold, the number-volume correspondence is that the expectation value of the number of elements in an order interval $\langle N\rangle$ is directly proportional to the volume V of the region of the manifold into which the interval embeds

从本质上来说，因果集理论不仅是时空的离散模型，也是时空的随机模型。因此，我们应当将序区间 N 的基数看作一个涨落量。为了方便起见，假设类流形因果集已被嵌入到一个流形中，那么数量-体积对应关系即指: 序区间 $\langle N\rangle$ 中元素数量的期望值，与该序区间所嵌入流形区域的体积 V 成正比

$$\langle N\rangle = \rho V \tag{2}$$

where ρ is the density of the embedded points in the region.

其中 ρ 是该区域内嵌入点的密度。

In addition to order intervals, two other subsets of C will be important to our discussion: chains and paths. A chain is a totally ordered subset of elements. That is, all pairs of elements are related. Let c_k denote a chain consisting of k elements, a k -chain:

除序区间外，C 的另外两个子集对我们的讨论十分重要: 链和路径。链是全序的元素子集，即所有元素对都满足序关系。令 c_k 表示由 k 个元素构成的链，即 k 链:

$$c_k \equiv \{e_1 \prec e_2 \prec \cdots \prec e_{k-1} \prec e_k\}. \tag{3}$$

A link $(\prec\cdot)$ is an irreducible relation between elements. If two elements are linked, there are no other elements between them: $e_i \prec \cdot e_k \Rightarrow \nexists e_j \ni e_i \prec e_j \prec e_k$. A path, therefore, is a chain consisting only of links. If p_k represents a k -path then

链接 $(\prec\cdot)$ 是元素之间的不可约序关系。如果两个元素由链接相连，则它们之间不存在其他元素: $e_i \prec \cdot e_k \Rightarrow \nexists e_j \ni e_i \prec e_j \prec e_k$ 。因此，路径就是仅由链接构成的链。若 p_k 表示一个 k 路径，则

$$p_k \equiv \{e_1 \prec \cdot e_2 \prec \cdots \prec \cdot e_{k-1} \prec \cdot e_k\}. \tag{4}$$

The length of a k -path is taken to be the number of links $(= k-1)$ within it.

k 路径的长度定义为其内部包含的链接数 $(= k-1)$ 。

In a spacetime manifold, the geodesic length between two events corresponds to the longest proper time between them. In a causal set, the length of the longest path between two elements is the most natural analog to the proper time. Again, assuming that a manifold-like causal set has been embedded into a manifold, the path-geodesic correspondence then is that in the large density limit, the length of the longest path between two related elements in a causal set becomes proportional to the proper time between the corresponding embedded events in the manifold. For an interval $I(e_i, e_k)$, the length of the longest path between its defining elements e_i and e_k is often called its height.

在时空流形中，两个事件之间的测地线长度对应它们之间的最长固有时。在因果集中，两个元素之间最长路径的长度是固有时最自然的类比。同样假设类流形因果集已被嵌入到一个流形中，那么路径-测地线对应关系即指: 在高密度极限下，因果集中两个有有序关系的元素之间最长路径的长度，与流形中对应嵌入事件之间的固有时成正比。对于一个区间 $I(e_i, e_k)$ ，其定义元素 e_i 和 e_k 之间最长路径的长度通常被称为该区间的高度。

Random Sprinklings

随机撒播

Recall that a manifold-like causal set is one that can be uniformly embedded into a manifold. One sure way to get a manifold-like causal set is to do the reverse and construct one from a uniform distribution of events in a spacetime manifold. This process is referred to as sprinkling.

回顾可知，类流形因果集是可均匀嵌入流形中的因果集。得到类流形因果集的一种可靠方法是反向操作，从时空流形中均匀分布的事件来构建它，这个过程被称为撒播。

Causal sets consistent with the properties stated previously can be formed by sprinkling points in a manifold via a Poisson process. In this process, the Poisson distribution is used to determine the number of events selected from a region of the manifold of volume V with the mean of the distribution determined by Eq. 2. This number of events is then uniformly selected within that region. The selected events are then identified with the elements of the causal set. The partial ordering of the elements is taken to coincide with the causal ordering of the events. After this, one has a bare, manifold-like causal set.

我们可以通过泊松过程向流形中撒播点，得到满足前文所述性质的因果集。该过程利用泊松分布确定从体积为 V 的流形区域中选取的事件数量，分布的均值由式 (2) 确定，确定数量后再在该区域内均匀选出这些事件，选中的事件即为因果集的元素，元素的偏序与事件的因果序一致，完成这些步骤后就得到了一个基础的类流形因果集。

To set some terminology, consider an order interval $I_A(e_i, e_k)$ formed by sprinkling into a region A of a manifold $\mathcal{M}$. Let (x, y) be the events in $\mathcal{M}$ that map to the elements (e_i, e_k) in I_A . The region A is the intersection of the causal future of x and the causal past of y : $A = \{\text{Future}(x) \cap \text{Past}(y)\}$. This region is often called an Alexandrov neighborhood or causal diamond. Therefore, order intervals in manifold-like causal sets map to Alexandrov neighborhoods in the manifold. Similarly, sprinkling an Alexandrov neighborhood generates causal sets that are order intervals (Sometimes the set of selected events in the manifold is referred to as an Alexandrov set.). It is on such causal sets that the manifold dimension estimators we will discuss have been tested.

为明确术语，考虑向流形 $\mathcal{M}$ 的区域 A 撒播得到的序区间 $I_A(e_i, e_k)$ 。设 (x, y) 为 $\mathcal{M}$ 中映射到序区间 I_A 内元素 (e_i, e_k) 的事件。区域 A 是 x 的因果未来与 y : $A = \{\text{未来}(x) \cap \text{Past}(y)\}$ 的因果过去的交集，这个区域通常被称为亚历山德罗夫邻域或因果菱形。因此，类流形因果集的序区间映射到流形中的亚历山德罗夫邻域，相应地，对亚历山德罗夫邻域撒播会得到作为序区间的因果集 (有时流形中被选中的事件集合也被称为亚历山德罗夫集)。我们后文讨论的流形维数估计量正是在这类因果集上完成了测试。

Myrheim-Meyer Dimension Estimators

迈尔海姆-迈尔维度估计量

In an important technical report, J. Myrheim [2] noted that for causal sets (Myrheim did not use the term causal sets.), a statistically derived notion of dimension is possible. As an example, he pointed out that for an order interval selected from Minkowski space, the ratio of the number of ordered pairs (2-chains) to the total number of pairs, which he called the ordering fraction, is a function of the dimension suggesting that this ratio could serve as a dimension estimator. Subsequently, David Meyer worked out the mathematical details of Myrheim's suggestion and placed it in the broader context of k -chain statistics [3]. The resulting dimension estimator, based on the statistics of 2-chains, has become known as the Myrheim-Meyer dimension.

J·迈尔海姆在一份重要技术报告中指出 [2]，对于因果集 (迈尔海姆当时未使用“因果集”这一术语)，可以通过统计推导得到维度的概念。他举例指出，对于从闵可夫斯基空间中选取的序区间，有序对 (2-链) 数量占总对数的比率 (他将其称为序分数) 是维度的函数，这意味着该比率可以用作维度估计量。随后，戴维·迈尔完善了迈尔海姆这一猜想的数学细节，并将其置于更广泛的 k -链统计背景下 [3]。由此得到的基于 2-链统计的维度估计量被称为迈尔海姆-迈尔维度。

This dimension estimator is a standard starting point for order-theoretic dimension estimation in causal set theory. It has spawned a whole class of dimension estimators, based on k -chain statistics, obtained through various modifications or extensions of Meyer's work in [3]. In this section, we will review some of the details of the Myrheim-Meyer dimension and a few of the modified Myrheim-Meyer techniques that have been introduced to either expand its use-cases or improve its accuracy. All of the dimension estimators we discuss assume manifold-like causal sets.

该维度估计量是因果集理论中序理论维度估计的标准起点。它衍生出了一整类基于 k -链统计的维度估计量，都是通过对迈尔在文献 [3] 中的工作进行各类修改或扩展得到的。在本节中，我们将介绍迈尔海姆-迈尔维度的部分细节，以及几种已提出的改进型迈尔海姆-迈尔方法，这些方法要么拓展了它的应用场景，要么提升了估计精度。我们讨论的所有维度估计量都适用于类流形因果集。

Myrheim-Meyer Dimension

Myrheim-Meyer 维数

In [3], Meyer worked out the expected number of k -chains in an embedded interval from a unit-density ($\rho = 1$) sprinkling in Minkowski space. He found

在文献 [3] 中，Meyer 推导出了闵氏空间中单位密度 k 撒播下嵌入区间内 ($\rho = 1$) 链的期望数量，得到结果如下

$$\langle c_k \rangle = \frac{V^k}{k}\left(\frac{\Gamma(d+1)}{2}\right)^{k-1} \frac{\Gamma(d/2)\,\Gamma(d)}{\Gamma(kd/2)\,\Gamma((k+1)\,d/2)}, \tag{5}$$

where d denotes the manifold dimension, V is the volume of the Alexandrov neighborhood, and $\Gamma(x)$ is the gamma function (In [3], d represents only the spatial dimension, whereas here it is the spacetime dimension. We have modified Meyer's expression to fit our usage.).

其中 d 表示流形维数，V 是亚历山德罗夫邻域的体积，$\Gamma(x)$ 是伽马函数 (在文献 [3] 中，d 仅代表空间维数，本文中 d 为时空维数，我们修改了 Meyer 的表达式以适配我们的用法。)

The most straightforward way to use this expression as a dimension estimator is to take $k = 2$,

将该表达式用作维数估计量最直接的方法是取 $k = 2$,

$$\langle c_2 \rangle = V^2 \frac{\Gamma(d+1)\,\Gamma(d/2)}{4\Gamma(3d/2)}, \tag{6}$$

and apply the following two facts: (a) For a unit-density sprinkling we can take $V = \langle N \rangle$ by Eq. 2. Note also that $\langle N \rangle = \langle c_1 \rangle$. (b) The expected number of 2-chains $\langle c_2 \rangle$, is just the expected number of relations $\prec$ in the causal set. Therefore, by simply counting the number of elements ($\approx V$) and the number of relations ($\approx \langle c_2 \rangle$) in the causal set, we obtain a ratio that depends solely on the dimension,

并应用以下两个结论:(a) 对于单位密度撒播，我们可以根据式 (2) 取 $V = \langle N \rangle$ 。另请注意 $\langle N \rangle = \langle c_1 \rangle$ 。(b) 2 链的期望数量 $\langle c_2 \rangle$ ，正是因果集合中关系数量的期望 $\prec$ 。因此，只需对因果集合中的元素数量 ($\approx V$) 和关系数量 ($\approx \langle c_2 \rangle$) 计数，我们就能得到一个仅依赖于维数的比值,

$$f(d) = \frac{\langle c_2 \rangle}{\langle c_1 \rangle^2} = \frac{\langle c_2 \rangle}{N^2} = \frac{\Gamma(d+1)\,\Gamma(d/2)}{4\Gamma(3d/2)}. \tag{7}$$

This function, $f(d)$, is half of Myrheim's ordering fraction and can be solved for the dimension numerically.

这个函数 $f(d)$ 是 Myrheim 定序分数的一半，可以通过数值方法求解得到维数。

Modified Myrheim-Meyer: Constant Cardinality Approach

修正迈尔海姆-迈耶: 恒定基数方法

Recently, Aghili et al. have proposed an alternative to the Myrheim-Meyer dimension [4]. In this approach, one seeks the manifold dimension (assuming Minkowski space) of a causal set by only considering subsets (sub-intervals) of fixed cardinality N . This choice presents two important differences from the development of the traditional Myrheim-Meyer dimension:

近期，阿吉里等人提出了迈尔海姆-迈耶维数的一种替代方案 [4]。在该方法中，人们仅通过考虑固定基数 N 的子集 (子区间)，来求解因果集的流形维数 (假设为闵可夫斯基空间)。该选择与传统迈尔海姆-迈耶维数的推导相比，存在两点重要差异:

1. For a uniform sprinkling at fixed N , the probability of finding a particular number of elements $n\,(< N)$ within a subregion of the Alexandrov neighborhood is governed by the binomial distribution rather than the Poisson distribution.

1. 对于固定 N 的均匀撒播，在亚历山德罗夫邻域的子区域内找到特定数量元素 $n\,(< N)$ 的概率服从二项分布，而非泊松分布。

2. Because N is fixed, the probability of getting a k -chain through a particular set of locations calls for an adjustment of the density for different factors in the probability.

2. 由于 N 固定，通过特定位置集合得到一条 k 链的概率需要对概率中不同因子的密度进行调整。

To explain the second point, we note that the probability of finding an element within a given infinitesimal volume is $dP = \rho dV$, where ρ is the density of elements. The probability of a k -chain through a particular set of infinitesimal volumes then is a product of $k-2$ such factors (assuming the endpoints are given). Once an element is located in some dV , there only remain $N-1$ elements left to locate within the entire volume so the effective density decreases. This decrease continues in subsequent factors until the result for the entire k -chain is obtained:

为解释第二点，我们注意到，在给定无穷小体积内找到一个元素的概率为 $dP=\rho dV$ ，其中 ρ 是元素密度。那么 (假设端点给定)，通过一组特定无穷小体积的 k 链的概率就是 $k-2$ 个此类因子的乘积。一旦某个元素定位于某 dV ，整个体积内待定位的元素就只剩 $N-1$ 个，因此有效密度会降低。这种降低会在后续因子中持续，直到得到整条 k 链的结果:

$$dP_k=\rho_0 dV_1\cdot\rho_1 dV_2\cdots\rho_{k-2}dV_{k-3}\cdot\rho_{k-1}dV_{k-2} \tag{8}$$

where the notation ρ_i indicates the density based on $N-i$ elements.

其中记号 ρ_i 表示基于 $N-i$ 个元素的密度。

Applying the two modifications, the expected number of k -chains becomes [5]

应用这两处修正后，k 链的期望数量变为 [5]

$$\langle c_k\rangle=\frac{N!}{(N-k)!}\left(\frac{\Gamma(d+1)}{2}\right)^{k-1}\frac{\Gamma(d/2+1)\,\Gamma(d)}{\Gamma(kd/2+1)\,\Gamma((k+1)\,d/2)} \tag{9}$$

This result can be compared with the Myrheim-Meyer result, Eq. 5, which we write in a slightly different form to facilitate comparison

该结果可与迈尔海姆-迈耶的结果即式 (5) 对比，我们将其改写为略有不同的形式以方便比较

$$\langle c_k\rangle_{\mathrm{MM}}=N^k\left(\frac{\Gamma(d+1)}{2}\right)^{k-1}\frac{\Gamma(d/2+1)\,\Gamma(d)}{\Gamma(kd/2+1)\,\Gamma((k+1)\,d/2)}. \tag{10}$$

As with the Myrheim-Meyer dimension, the simplest application of Eq. 9 as a dimension estimator is to take $k=2$. Doing this gives

和迈尔海姆-迈耶维数一样，将式 (9) 用作维数估计量的最简单应用是取 $k=2$ ，由此可得

$$\frac{\langle c_2\rangle}{N(N-1)}=\frac{\Gamma(d+1)\,\Gamma(d/2)}{4\Gamma(3d/2)}. \tag{11}$$

Again, counting the number of relations $\langle c_2\rangle$, leads to a ratio that only depends on d which can be solved numerically.

同样地，对关系 $\langle c_2\rangle$ 的数量进行计数，会得到一个仅依赖于 d 的比值，可通过数值方法求解。

Modified Myrheim-Meyer: Curved Spacetimes

修正米尔海姆-迈耶法: 弯曲时空

As discussed in section ”Myrheim-Meyer Dimension”, the Myrheim-Meyer dimension uses the ratio of 2-chains to 1-chains, Eq. 7, to estimate the flat spacetime manifold dimension of causal sets. While the choice $k = 2$ is easiest to apply, larger k -chains can be used. In the context of curved spacetimes, application of appropriately modified versions of Eq. 5 for multiple values of k has been used to obtain the manifold dimension and other quantities [3, 6, 7] .

正如“米尔海姆-迈耶维数”一节所讨论，米尔海姆-迈耶维数利用 2-链与 1-链的比值 (式 7) 估计因果集的平直时空流形维数。尽管选择 $k = 2$ 最易应用，也可使用更大的 k -链。在弯曲时空背景下，对多个 k 值应用适当修正后的式 5，已被用于获取流形维数及其他物理量 [3, 6, 7] 。

One method for using higher k -chains as a dimension estimator, suggested by Roy et al. [6] is to compare the distribution of the abundance of k -chains as a function of k in a causal set to the corresponding distribution as determined by Eq. 5. Agreement between these distributions would be an indicator of the manifold properties of the causal set. This approach was implemented by Kambor and X [7] who used Eq. 5 to compute $\langle c_k \rangle_\eta$ up to $k = 10$. The subscript η is intended to indicate that this is the Minkowski space result. Making the definition

罗伊等人 [6] 提出了一种将高阶 k -链用作维数估计器的方法: 将因果集中 k -链的丰度随 k 变化的分布，与式 5 给出的对应分布进行比较。两类分布的吻合可作为因果集具有流形性质的指示。坎博和 X[7] 实现了该方法，他们利用式 5 计算了直至 $k = 10$ 的 $\langle c_k \rangle_\eta$ 。下标 η 用于表明这是闵氏空间的结果。定义如下

$$\overline{\langle c_k \rangle} \equiv \frac{\langle c_k \rangle}{\langle c_1 \rangle^k}, \tag{12}$$

they calculated $\overline{\langle c_k \rangle}_\eta$ using Eq. 5 for $k = 1, 2, \cdots 10$. Their results are shown as the solid curves in Fig. 1a, which shows that these distributions clearly distinguish the different dimensions of Minkowski space.

他们利用式 5 对 $k = 1, 2, \cdots 10$ 计算了 $\overline{\langle c_k \rangle}_\eta$ 。结果如图 1a 中的实线所示，可见这些分布可以明确区分闵氏空间的不同维数。

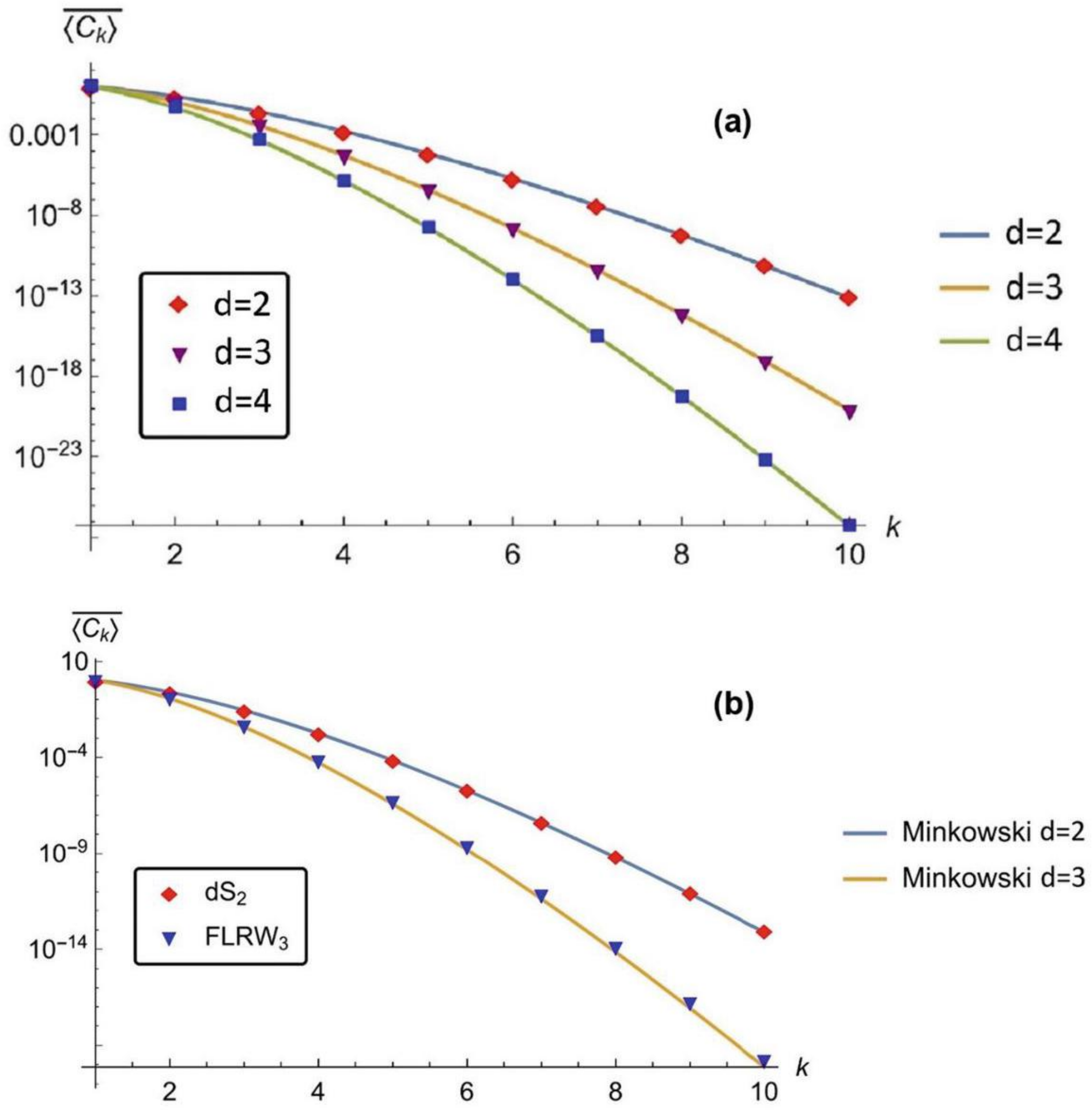


Fig. 1 The distribution of the abundance of k -chains as an indicator of manifold dimension [7]. Panel (a) is for Minkowski space. The data points are from simulations, and the solid lines are from Eq. 5. In panel (b), the data points are from simulations with dS_2 and $FLRW_3$, and the solid lines are for Minkowski space in 2 and 3 dimensions. (©IOP Publishing. Reproduced with permission. All rights reserved)

图 1 k -链的丰度分布作为流形维数的指示 [7]。(a) 图对应闵氏空间，数据点来自模拟，实线来自式 5。(b) 图中，数据点是含 dS_2 和 $FLRW_3$ 的模拟结果，实线是 2 维与 3 维闵氏空间的结果。(©IOP 出版社，经许可复制，版权所有)

To leverage these distributions as an indicator of the manifold dimension of curved spacetimes, they formed order intervals from random sprinklings in two curved spacetimes: two-dimensional de Sitter space and three-dimensional FLRW (FLRW stands for Friedmann-Lemaître-Robertson-Walker.). Figure 1b shows the comparison of the order-theoretic calculations of $\overline{\langle c_k \rangle}$ in those curved spacetime intervals against the Minkowski space distributions. The good agreement between the distributions from intervals obtained from curved spacetimes and the theoretical distributions from the flat spacetimes of like dimension suggests that this method can serve as a useful indicator of the dimension of curved manifold-like causal sets.

> 为利用这类分布指示弯曲时空的流形维数，他们从两种弯曲时空的随机撒布构造了序区间: 二维德西特空间与三维 FLRW 空间 (FLRW 是弗里德曼-勒梅特-罗伯逊-沃尔克的缩写)。图 1b 对比了这些弯曲时空序区间中 $\overline{\langle c_k \rangle}$ 的序理论计算结果与闵氏空间分布。来自弯曲时空的区间分布与相同维数平直空间的理论分布吻合良好，说明该方法可有效指示类弯曲流形因果集的维数。

Another method using multiple k -chains, worked out by Roy et al. [6], uses a system of k -chains to study dimension estimation and estimation of curvature. They focused on order intervals generated from small causal diamonds in curved spacetimes so that only low-order curvature corrections would be important. Consistent with this assumption, the authors used Riemann normal coordinates (RNC) so that the metric could be represented as

> 罗伊等人 [6] 还提出了另一种使用多 k -链的方法，该方法利用 k -链体系研究维数估计与曲率估计。他们聚焦弯曲时空中小因果钻石生成的序区间，因此仅需考虑低阶曲率修正。符合该假设，作者使用黎曼法坐标 (RNC) 将度规表示为

$$g_{ab}(x) = \eta_{ab}(0) - \frac{1}{3} x^c x^d R_{abcd}(0) + O\left(x^3\right), \tag{13}$$

where $\eta_{ab}(0)$ and $R_{abcd}(0)$ represent components of the Minkowski metric and the Riemann tensor at some chosen origin, respectively. This approximation was applied to an Alexandrov neighborhood $A[p,q]$ with τ , the proper time from p to q , used as the expansion parameter by which the neighborhood $A[p,q]$ was kept small.

> 其中 $\eta_{ab}(0)$ 和 $R_{abcd}(0)$ 分别代表所选原点处闵氏度规与黎曼张量的分量。该近似被应用于亚历山大罗夫邻域 $A[p,q]$ ，将从 p 到 q 的固有时 τ 作为膨胀参数，以此保证邻域 $A[p,q]$ 足够小。

Performing an analysis similar to [3], they obtained the lowest-order curvature corrections to $\langle c_k \rangle$ for a sprinkling of events in $A[p,q]$,

> 通过开展与文献 [3] 类似的分析，他们得到了 $A[p,q]$ 中事件撒布的 $\langle c_k \rangle$ 最低阶曲率修正，

$$\langle c_k \rangle = \langle c_k \rangle_\eta \left[1 + (\alpha_k R + \beta_k R_{00})\,\tau^2\right] + O\left(\tau^{kd+3}\right). \tag{14}$$

In this expression, R and R_{00} are the scalar curvature and the time-time component of the Ricci tensor at the origin. The coefficients α_k and β_k depend on the dimension and are given by

> 该式中，R 和 R_{00} 分别是原点处的标量曲率和里奇张量的时-时分量。系数 α_k 和 β_k 依赖于维数，由下式给出

$$\alpha_k = \frac{kd}{12\,(kd+2)\,((k+1)\,d+2)}, \tag{15}$$

and

和

$$\beta_k = \frac{kd}{12\left(\left(k+1\right)d+2\right)}. \tag{16}$$

Equations 14-16 were obtained for randomly selected events in the manifold. We now apply these expressions to the order interval $I_A\left[e_i, e_j\right]$ that maps to the Alexandrov neighborhood $A\left[p, q\right]$. In this reinterpretation, the proper time is replaced by the height of the interval T . For a manifold-like causal set, the height T then serves as an estimator for the proper time. In fact, Eq. 14 contains four such quantities. In addition to T , there are manifold estimators for the scalar curvature R , for R_{00} , and for the dimension d .

方程 14-16 是针对流形中随机选取的事件推导得到的。我们现在将这些表达式应用于映射到亚历山德罗夫邻域 $A\left[p, q\right]$ 的序区间 $I_A\left[e_i, e_j\right]$ 。在该重新诠释中，固有时被替换为区间 T 的高度。对于类流形因果集，高度 T 可作为固有时的估计量。事实上，式 14 包含四个此类量。除 T 外，还有标量曲率 R 、R_{00} 以及维数 d 的流形估计量。

Therefore, a system of four equations is constructed from the $\langle c_k\rangle$ for $k = 1, \ldots, 4$, which can be algebraically manipulated to eliminate R, R_{00} , and T . This leaves only d and the $\langle c_k\rangle$. As with the other dimension estimators of this type, the $\langle c_k\rangle$ can simply be counted for the causal set in question leaving an expression in terms of d that can be solved numerically.

因此，我们由 $\langle c_k\rangle$ 构造出关于 $k = 1, \ldots, 4$ 的四元方程组，可通过代数操作消去 R, R_{00} 和 T ，最终仅保留 d 和 $\langle c_k\rangle$。与该类型的其他维数估计方法一样，只需对目标因果集计数得到 $\langle c_k\rangle$，即可得到仅含 d 的表达式，该表达式可通过数值方法求解。

Carrying out this prescription, the authors obtained the following equation;

遵循该流程，作者得到了以下方程:

$$\sum_{k=1}^{4} \omega_k\left(d\right)\langle c_k\rangle^{4/k} = 0, \tag{17}$$

where

其中

$$\omega_k\left(d\right) \equiv \left(-1\right)^{k-1}\binom{3}{k-1}\frac{\left(kd+2\right)\left[\left(k+1\right)d+2\right]}{\chi_k^{4/k}}, \tag{18}$$

with

满足

$$\chi_k \equiv \frac{1}{k}\left(\frac{\Gamma\left(d+1\right)}{2}\right)^{k-1}\frac{\Gamma\left(d/2\right)\Gamma\left(d\right)}{\Gamma\left(kd/2\right)\Gamma\left(\left(k+1\right)d/2\right)} \tag{19}$$

It is this equation that is solved for the dimension using only order-theoretic quantities as inputs.

只需输入序理论量即可求解该方程得到维数。

As noted in [6], there is nothing special about the choice to use $k = 1, \ldots, 4$ to generate the four equations needed, any 4 will suffice. Kambor and X implemented this generalization to larger k -chains [7]. Their result is expressed in the following way:

正如文献 [6] 指出，选择 $k = 1, \ldots, 4$ 生成所需的四个方程并非强制，任意四个都满足要求。Kambor 和 X 将该方法推广到更大的 k 链 [7]，他们的结果表述如下:

$$\sum_{l=0}^{3} (-1)^l \binom{3}{l} j_{k_1+l}(d) \left[\frac{\langle c_{k_1+l}\rangle}{\chi_{k_1+l}}\right]^{\mu/(k_1+l)} = 0, \tag{20}$$

where

其中

$$j_k(d) \equiv (kd+2)\left[(k+1)d+2\right]. \tag{21}$$

In Eq. 20, $k_1 \geq 1$ is the starting value such that four consecutive integer values are used in the summation.

在式 20 中，$k_1 \geq 1$ 是起始值，求和时使用四个连续整数值。

The authors also explored the use of the free parameter μ , where taking $k_1 = 1$ and $\mu = 4$ recovers the original result of Roy et al., Eq. 17. They showed that for the 3-dimensional FLRW spacetime, the original estimator, $(k_1, \mu) = (1, 4)$, exhibits multiple zeros. Therefore, a numerical solution via a root-finding algorithm could return the wrong root. They found that taking $\mu < 1$ removed this ambiguity and still gave an accurate estimate of the manifold dimension.

作者还研究了自由参数 μ 的使用，当取 $k_1 = 1$ 和 $\mu = 4$ 时，可还原 Roy 等人的原始结果即式 17。他们指出，对于三维 FLRW 时空，原始估计量 $(k_1, \mu) = (1, 4)$ 存在多个零点，因此求根算法得到的数值解可能是错误根。他们发现取 $\mu < 1$ 可消除该歧义，同时仍能给出准确的流形维数估计。

Other Order-Theoretic Dimension Estimators

其他序论维度估计量

In this section, we discuss several order-theoretic methods not based on k -chain statistics in the way that the Myrheim-Meyer dimension, and modifications thereof, are. These dimension estimators are all specific to Minkowski space and all leverage the known relationship for how the volume of an Alexandrov neighborhood scales with proper time.

本节我们讨论若干序论方法，这些方法并不像米尔海姆-迈耶维度及其修正版本那样基于 k 链统计。所有这些维度估计量都专门针对闵可夫斯基空间，且都利用了已知关系，即亚历山德罗夫邻域的体积如何随固有时缩放。

Let p and q be the defining events of an Alexandrov neighborhood in Minkowski space with $q \in \text{Future}\,(p)$ and let V be its volume. This volume is given by

设 p 和 q 是闵可夫斯基空间中 $q \in \text{Future}\,(p)$ 条件下一个亚历山德罗夫邻域的定义事件，V 为该邻域的体积。该体积可表示为

$$V = \frac{\pi^{(d-1)/2}}{2^{d-2} d\,(d-1)\,\Gamma\left[(d-1)/2\right]} \tau^d \tag{22}$$

where τ is the proper time from p to q , and d is the dimension of the spacetime.

其中 τ 是从 p 到 q 的固有时，d 是时空维度。

Midpoint Scaling Method

中点标度法

The midpoint scaling dimension, proposed by Bombelli [8], is based on both the number-volume and path-geodesic correspondences. Given an order interval $I\left(e_i, e_f\right)$ of cardinality N , the midpoint dimension can be obtained as follows. An element e_m in I defines two subintervals, $I_1\left(e_i, e_m\right)$ and $I_2\left(p_m, e_f\right)$. Choose e_m such that the cardinality N_m of the smaller subinterval I_m is as large as possible. If I can be uniformly embedded in Minkowski space, the height of I_m will be about half that of I . In the manifold, consider the corresponding Alexandrov neighborhoods $A \mapsto I$ and $A_m \mapsto I_m$ with V and V_m as their volumes, τ and τ_m as their longest proper times. By construction, $\tau_m = \tau/2$. Therefore, Eq. 22 gives $V_m = V/2^d$.

邦贝利提出的中点标度维数 [8] 同时基于数-体积对应关系与路径-测地线对应关系。给定一个基数为 N 的序区间 $I\left(e_i, e_f\right)$ ，可按如下步骤得到中点维数。I 中的元素 e_m 定义了两个子区间 $I_1\left(e_i, e_m\right)$ 和 $I_2\left(p_m, e_f\right)$。选择 e_m ，使得较小子区间 I_m 的基数 N_m 取最大值。若 I 能一致嵌入闵可夫斯基空间，则 I_m 的高度约为 I 的一半。在流形中，考虑对应的亚历山大罗夫邻域 $A \mapsto I$ 和 $A_m \mapsto I_m$ ，其体积分别为 V 和 V_m ，固有最长时分别为 τ 和 τ_m 。根据构造可得 $\tau_m = \tau/2$ 。因此由式 (22) 可得 $V_m = V/2^d$ 。

By the path-geodesic correspondence, the heights of I and I_m scale as τ and τ_m , respectively. The number-volume correspondence also tells us that N and N_m scale as V and V_m . Therefore, the relationship between N and N_m should follow that of the volumes: $N_m = N/2^d$. From this relation, we obtain the midpoint dimension:

根据路径-测地线对应关系，I 和 I_m 的高度分别按 τ 和 τ_m 标度。根据数-体积对应关系也可知，N 和 N_m 分别按 V 和 V_m 标度。因此 N 和 N_m 的关系应满足体积的标度关系: $N_m = N/2^d$ 。由此关系可推导出中点维数:

$$d = \frac{\ln\left(N/N_m\right)}{\ln\left(2\right)} \tag{23}$$

Brightwell-Gregory Method

Brightwell-Gregory 方法

Brightwell and Gregory [9] proposed a method, which, like the midpoint scaling method, depends on how the proper time between causally connected events in Minkowski space scales with the volume of the Alexandrov neighborhood they define. This method is based on a result, previously established in [10], that for an order interval of height T generated by sprinkling at density ρ into an Alexandrov neighborhood of volume V in Minkowski space, with high probability

Brightwell 和 Gregory[9] 提出了一种方法，该方法与中点标度法类似，依赖于闵氏空间中因果关联事件之间的固有时随这两个事件定义的亚历山德罗夫邻域体积的标度关系。该方法基于此前文献 [10] 中得到的一个结论: 对于以密度 ρ 撒入闵氏空间中体积为 V 的亚历山德罗夫邻域、高度为 T 的序区间，高概率满足

$$\lim_{\rho\to\infty} \frac{T}{(\rho V)^{1/d}} = m_d \tag{24}$$

where m_d is a dimension dependent constant near the value 2 .

其中 m_d 是依赖于维度的常数，取值接近 2。

Therefore, given a sufficiently large manifold-like interval of height T , we can estimate the dimension by adopting $m_d = 2$, taking $\rho V = N$, where N is the cardinality of the interval, and turning Eq. 24 into an equality: $N^{1/d} = T/2$. The estimate of the manifold dimension then is

因此，给定一个高度为 T 、足够大的类流形区间，我们可以通过引入 $m_d = 2$ ，取 $\rho V = N$ (其中 N 是该区间的基数)，将式 (24) 转化为等式: $N^{1/d} = T/2$ ，从而估计维度。流形维度的估计结果为

$$d = \frac{\ln N}{\ln(T/2)} \tag{25}$$

Average Path Length Method

平均路径长度法

Aghili et al. [11] proposed a modification of the Brightwell-Gregory method using the average path length between the minimal and maximal elements of an order interval rather than its height (the maximum path length). The rationale for this modification is that in the stochastic environment of causal sets, the mean path length would be subject to less statistical noise than the maximum path length.

Aghili 等人 [11] 提出了对 Brightwell-Gregory 方法的改进，该方法使用序区间最小元与最大元之间的平均路径长度，而非序区间的高度 (即最大路径长度)。这一改进的理论依据是，在因果集的随机环境中，平均路径长度受到的统计噪声小于最大路径长度。

To implement this method, we first define the (irreflexive) link matrix

要实现该方法，我们首先定义 (非自反的) 链接矩阵

$$L_{ij} = \begin{cases} 1, & e_i \prec e_j \\ 0, & \text{otherwise} \end{cases} \tag{26}$$

which is one of several ways to represent a causal set. The number of k -element paths between two elements e_i and e_j can be found by taking k products of the link matrix. So, for an interval $I\left[e_i, e_j\right]$, the number of k -paths between the minimal and maximal elements is given by

它是表示因果集的几种方式之一。两个元素 e_i 和 e_j 之间的 k 元路径数量可以通过对链接矩阵做 k 次乘积得到。因此，对于一个区间 $I\left[e_i, e_j\right]$ ，其最小元和最大元之间的 k 路径数量可表示为

$$n_k = \left(L^k\right)_{ij} \tag{27}$$

and the total number of paths between those elements is

而这些元素之间的路径总数为

$$N_{\text{paths}} = \sum_{k=1}^{N} n_k \tag{28}$$

Using these quantities, the average path length is simply

利用这些量，平均路径长度可简单表示为

$$\langle \ell \rangle = \frac{1}{N_{\text{paths}}} \sum_{k=1}^{N} (k-1)\, n_k. \tag{29}$$

As with T in the Brightwell-Gregory method, as N becomes large $\langle \ell \rangle$ will become proportional to $N^{1/d}$ except with a different constant of proportionality. So we can write a relation analogous to Eq. 24,

和 Brightwell-Gregory 方法中的 T 类似，当 N 变得很大时，$\langle \ell \rangle$ 会与 $N^{1/d}$ 成正比，只是比例常数不同。因此我们可以写出类似于式 (24) 的关系:

$$\lim_{N\to\infty} \frac{\langle \ell \rangle}{N^{1/d}} = \alpha_d \tag{30}$$

where α_d is a d -dependent constant that can be determined using simulations. The dimension estimator then is

其中 α_d 是依赖于 d 的常数，可以通过模拟确定。维度估计量为

$$d = \frac{\ln N}{\ln\left(\langle \ell \rangle / \alpha_d\right)}. \tag{31}$$

Aghili et al. found that $\alpha_d \approx 1.15$ worked well for all dimensions.

Aghili 等人发现，$\alpha_d \approx 1.15$ 适用于所有维度的估计。

Spectral Dimension Estimators

谱维度估计量

The spectral dimension provides an alternative to the order-theoretic approaches discussed in sections "Myrheim-Meyer Dimension Estimators" and "Other Order-Theoretic Dimension Estimators". Traditionally, the spectral dimension of a space is based on the fact that the probability for a random walker to return to its original position depends on the dimension of the space in which the random walk takes place. The walk can be described in terms of a diffusion process tracked by a parameter σ , called the diffusion time, representing the number of steps of the random walk. If we denote the return probability as a function of the diffusion time as $P_{\mathrm{r}}(\sigma)$ then, in a flat space, this probability scales as $P_{\mathrm{r}}(\sigma) \approx \sigma^{-d_{\mathrm{s}}/2}$, where d_{s} is the dimension of the space. Therefore, we determine the dimension by taking the following derivative

谱维度是“迈尔海姆-迈耶维度估计量”和“其他序理论维度估计量”两节中讨论的序理论方法的替代方案。传统上，空间的谱维度基于如下原理: 随机游走者返回起始位置的概率取决于随机游走所在空间的维度。游走可以用扩散过程描述，由参数 σ 跟踪，该参数称为扩散时间，代表随机游走的步数。若我们将返回概率记为扩散时间的函数 $P_{\mathrm{r}}(\sigma)$ ，那么在平坦空间中，该概率满足标度关系 $P_{\mathrm{r}}(\sigma) \approx \sigma^{-d_{\mathrm{s}}/2}$ ，其中 d_{s} 是空间的维度。因此我们可以通过如下求导得到维度

$$d_{\mathrm{s}} = -\frac{d}{d\ln\sigma}\ln P_{\mathrm{r}}(\sigma)\,. \tag{32}$$

This value is known as the spectral dimension (From the point-of-view of diffusion governed by the diffusion equation, $\partial P/\partial\sigma = \nabla^2 P$, the return probability can also be written in terms of the eigenvalue spectrum of the ∇^2 operator, which is suggestive of the name spectral dimension.). Figure 2 depicts such a random walk on a discrete set of points generated by a random sprinkling in a flat space.

该值就是谱维度 (从由扩散方程控制的扩散过程角度来看，$\partial P/\partial\sigma = \nabla^2 P$ ，返回概率也可以通过 ∇^2 算符的特征谱写出，“谱维度”的名称也由此而来。) 图 2 展示了平坦空间中随机撒点生成的离散点集上的这类随机游走。

Spectral Dimension of Causal Sets

因果集的谱维数

The spectral dimension was introduced to causal set theory by Eichhorn and Mizera [13], who studied the behavior of the spectral dimension for causal sets generated in various ways including by sprinklings in different spacetime manifolds. As discussed in [13], the Lorentzian nature of spacetime presents challenges for a random walk not relevant to random walks in a purely spatial manifold. Because elements in a causal set represent locations in both space and time, one conceptual challenge stems from the fact that the probability of return to the original location in the causal set does not just represent a return to the original point in space, but also a return to an earlier time, which violates causality. This violation of causality is merely a conceptual problem because in a computer simulation the fictitious random walker can still be allowed to return to the original position in the causal set.

谱维数由艾希霍恩和米泽拉 [13] 引入因果集理论，他们研究了通过多种方式 (包括在不同时空流形中撒播) 生成的因果集的谱维数的行为。正如文献 [13] 所述，时空的洛伦兹属性给随机游走带来了纯空间流形中的随机游走不会面临的挑战。由于因果集中的元素同时代表空间和时间中的位置，一个概念层面的难题源于: 返回因果集中初始位置的概率不仅代表返回空间中初始点，还代表返回更早的时间，这违反了因果性。这种因果性违反只是概念层面的问题，因为在计算机模拟中，依然可以允许虚构的随机游走者返回因果集中的初始位置。

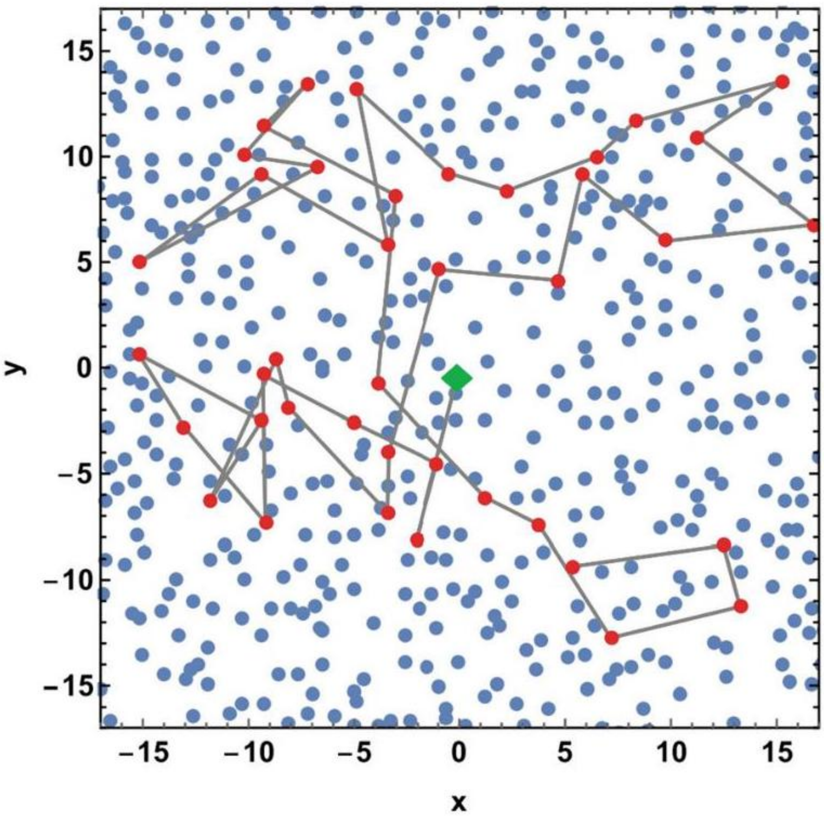
To remove the noncausal aspects of the spectral dimension as defined by Eq. 32, [13] also defines a form of the spectral dimension, referred to as the causal spectral dimension d_{cs} , based on the meeting probability, $P_{\mathrm{m}}(\sigma)$, of two independent random walkers who are restricted to only take steps forward in time. This dimension estimator is determined from the meeting probability similarly as d_{s} is from the return probability,

为消除式 (32) 定义的谱维数中的非因果方面，文献 [13] 还定义了一种谱维数形式，称为因果谱维数 d_{cs} ，它基于两个只能沿时间正向走步的独立随机游走者的相遇概率 $P_{\mathrm{m}}(\sigma)$ 。该维数估计量由相遇概率确定，其方式和 d_{s} 由返回概率确定类似，

$$d_{\mathrm{cs}} = -\frac{d}{d\ln\sigma}\ln P_{\mathrm{m}}(\sigma)\,. \tag{33}$$

Fig. 2 A visual example of a random walk used to determine the spectral dimension [12]. The central diamond at the origin is the starting point and the dots are points selected by sprinkling. The straight lines represent one particular diffusion path. (C) IOP Publishing. Reproduced with permission. All rights reserved)

图 2 用于确定谱维数的随机游走的直观示例 [12]。原点处的中心菱形是起点，点是撒播选出的点。直线代表一条特定的扩散路径。(C) IOP 出版公司。经许可可重印。版权所有)



The key finding from [13] relevant to our purpose is that for the manifold-like causal sets studied, both spectral dimension estimators, d_s and d_{cs} , were shown to approach the manifold dimension for sufficiently large diffusion times. Furthermore, when these methods were applied to non-manifold-like causal sets, they appropriately failed to produce results that might be misinterpreted as a manifold dimension, thereby avoiding any false positives. In fact, for these non-manifold-like causal sets, both d_s and d_{cs} were found to approach zero for large diffusion times. These results offer evidence in support of these spectral dimensions as useful estimators for the manifold dimension of causal sets.

对于我们的研究而言，文献 [13] 的核心结论是: 在所研究的类流形因果集中，当扩散时间足够长时，两个谱维数估计量 d_s 和 d_{cs} 都会趋近于流形维数。此外，将这些方法应用于非类流形因果集时，它们不会给出可能被误读为流形维数的结果，从而避免了所有假阳性。实际上，对于这类非类流形因果集，研究发现当扩散时间很长时，d_s 和 d_{cs} 都会趋近于零。这些结果证明，这些谱维数是估计因果集流形维数的有用工具。

Spectral Dimension of Spatial Hypersurfaces

空间超曲面的谱维数

In addition to the violation of causality mentioned previously, calculating the spectral dimension, d_s , on the full causal set also faces the challenge of how to deal with the nonlocality inherent in the causal set model [14]. For the random walker, this nonlocality effectively means that each element in the set has an infinite number of nearest neighbors (see [13] for discussion). To bypass this challenge, in [12] Eichhorn and coworkers developed a scheme for determining the spectral dimension on the causal set equivalent of a spatial hypersurface, an inextendible antichain.

除了前文提到的因果性破缺问题，在整个因果集上计算谱维数 d_s 还面临如何处理因果集模型固有的非定域性问题 [14]。对随机行走者而言，这种非定域性实际上意味着集合中的每个元素都有无穷多个近邻 (相关讨论见 [13])。为绕过这一难题，艾希霍恩 (Eichhorn) 及其同事在文献 [12] 中提出了一种方案，在因果集对应空间超曲面的不可延拓反链上求解谱维数。

An antichain is a subset of a causal set consisting of mutually unrelated elements. If the full causal set is embedded in a spacetime manifold, the events obtained from the elements of an antichain would all be space-like related to each other. Subsets of these events would lie in a spatial hypersurface of dimension $d-1$, where d is the dimension of the spacetime manifold. This procedure seeks to identify the dimension of the hypersurface.

反链是因果集的一个子集，由互不相关的元素构成。若整个因果集嵌入一个时空流形，则反链所有元素对应的事件彼此都是类空关联的。这些事件的子集将位于维度为 $d-1$ 的空间超曲面上，其中 d 是时空流形的维度。该方法的目标就是求解这个超曲面的维度。

Eichhorn and coworkers performed the calculations to determine the spectral dimension on certain antichains of causal sets generated from sprinklings in three-dimensional Minkowski space. They found that at intermediate diffusion times, and on intermediate size scales, this spectral dimension does agree with the manifold dimension of a spatial hypersurface of three-dimensional Minkowski space from which the causal set was generated, i.e., $d_s \approx 2$. Therefore, this approach, which not only circumvents issues of locality but also of causality violations, offers a promising alternative for calculating (or providing a consistency check on) the manifold dimension.

艾希霍恩及其同事在三维闵氏空间撒播生成的因果集的特定反链上计算了谱维数。他们发现，在中间扩散时间和中间尺度下，该谱维数与生成因果集的三维闵氏空间空间超曲面的流形维数一致，即等于 $d_s \approx 2$。因此，该方法不仅避开了定域性问题，也解决了因果性破缺的问题，为计算 (或验证) 流形维数提供了一种很有前景的替代方案。

Beyond the Manifold Dimension

超出流形维度

While the focus of this chapter is on ways to estimate the manifold dimension as we have defined it in section ”Introduction”, the study of the dimension as a physical quantity has considerable value in its own right beyond its use as input for tests of macroscopic manifold-like behavior. This point is nicely summarized for causal sets and other approaches to quantum gravity in a review article by Steven Carlip [15].

尽管本章的重点是我们在“引言”一节中定义的流形维度估计方法，但无论如何，将维度作为物理量进行研究本身，除了用作宏观类流形行为检验的输入量之外，就具备相当可观的研究价值。史蒂文·卡利普在一篇综述文章中很好地总结了这一点，内容涵盖因果集与其他量子引力研究方法 [15]。

In this section, we draw attention to two phenomena of current interest-dimensional reduction and asymptotic silence-for which dimension estimation that does not seek a macroscopic manifold dimension can shed some light. Both topics are a consequence of the realization across different approaches to quantum gravity that the effective dimension of spacetime is dynamical and scale-dependent [15]. So how then, does the dimension behave on different scales in causal set theory?

在本节中，我们将关注目前受到研究关注的两个现象——维数约化与渐近沉默——不寻求宏观流形维数的维数估计可以为这两个现象提供一些启发。这两个课题都是不同量子引力研究方法得到的共同结论的产物: 时空的有效维数是动力学的，且随尺度变化 [15]。那么，在因果集合论中，维数在不同尺度上的表现究竟如何呢?

Dimensional Reduction

维度约化

The term dimensional reduction has different meanings in different contexts. Here, it refers to the fact that over the last couple of decades several approaches to quantum gravity appear to display a common feature that the dimension of spacetime reduces to a value near 2 on small scales. This realization is very enticing because such commonalities are rare. At the time of this writing, the extent to which this phenomenon is observed in causal sets is still an open question.

维度约化这个术语在不同语境下有不同含义。本文中它指的是: 过去几十年里，多个量子引力研究方法似乎都呈现出一个共同特征，即小尺度下的时空维度会降至接近 2 的值。这一发现十分吸引人，因为这种共性十分罕见。截至撰写本文时，这种现象在因果集中被观测到的程度仍然是一个悬而未决的问题。

To get a sense of why this remains controversial in causal sets, let us briefly address the calculation of the dimension on small scales. As can be seen in sections ”Myrheim-Meyer Dimension Estimators” and ”Spectral Dimension Estimators”, there are many ways to estimate the dimension, and these may not always agree in different contexts. The spectral dimension has emerged as a common approach in this regard. It has wide-range applicability because regardless of how an approach models spacetime, as long as there is a form of spacetime, one can devise a random walk in that structure.

为了说明为什么它在因果集理论中仍存在争议，我们来简要谈谈小尺度下的维度计算。正如在“迈尔海姆-迈耶维度估计量”和“谱维度估计量”章节中所见，维度估计有多种方法，且这些方法在不同语境下不一定总能给出一致结果。谱维度已经成为这一方向的常用方法。它适用范围极广，因为无论一种方法如何建模时空，只要存在某种形式的时空，就可以在该结构中设计随机游走。

In section ”Spectral Dimension Estimators”, we highlighted use of the spectral dimension to estimate the manifold dimension. However, the primary interest in the spectral dimension of causal sets is to probe these dynamical and scale-dependent issues. For example, the short, intermediate, and long diffusion time regimes of the random walk probe the spacetime structure on corresponding length scales. Also, several of the order-theoretic estimators, such as the midpoint scaling dimension, rely on macroscopic correspondences between causal sets and spacetime manifolds, such as the path-geodesic correspondence. Such correspondences are likely to break down on the smallest scales. The meaning of the spectral dimension is less strongly tied to macroscopic concepts.

在“谱维度估计量”章节中，我们重点介绍了用谱维度估计流形维度的方法。但研究因果集谱维度的主要目的是探究这些与动力学相关、依赖尺度的问题。例如，随机游走的短、中、长扩散时间区域对应探测不同长度尺度的时空结构。此外，几种序理论估计量 (比如中点标度维度) 依赖于因果集和时空流形之间的宏观对应关系，例如路径-测地线对应。这种对应关系在最小尺度下很可能失效，而谱维度的含义没有和宏观概念绑定得那么紧密。

Eichhorn and Mizera [13], in addition to investigating the longer diffusion times discussed previously, also studied the small-scale behavior of the spectral dimension of causal sets. They found that on the full causal set, both the spectral dimension and the causal spectral dimension showed a dimensional increase on small scales. This finding put causal set theory in contention with the several other approaches that show a decrease to two dimensions.

艾希霍恩和米泽拉 [13] 除了研究前文讨论的长扩散时间情况，还研究了因果集谱维度的小尺度行为。他们发现，在完整因果集上，谱维度和因果谱维度在小尺度下都呈现维度上升。这一结果让因果集理论和其他多种呈现维度降至 2 的研究方法产生了分歧。

However, in 2016, Belenchia et al. [16] calculated the spectral dimension of causal sets using different operators than used in [13], which they argued to be better suited to causal sets. Their results do show dimensional reduction to $d_s = 2$ on small scales. Furthermore, the spectral dimension on a spatial hypersurface discussed in section ”Spectral Dimension of Spatial Hypersurfaces” was also used to probe the small-scale behavior [12]. For this nonrelativistic version of the spectral dimension, Eichhorn and coworkers observed dimensional reduction, although not to a value of 2 .

但在 2016 年，贝伦基亚等人 [16] 使用了不同于文献 [13] 的算符计算因果集的谱维度，他们认为这些算符更适配因果集。他们的结果确实显示小尺度下维度约化为 $d_s = 2$ 。此外，“空间超曲面的谱维度”章节讨论的空间超曲面上的谱维度也被用于探测小尺度行为 [12]。对于这种非相对论版本的谱维度，艾希霍恩及其合作者观测到了维度约化，只是没有降到 2。

Even though the spectral dimension has emerged as a preferred method for investigating dimensional reduction, it remains interesting to explore the results of other dimension estimators. Abajian and Carlip used the Myrheim-Meyer dimension to study small subsets of causal sets faithfully embeddable in 3-, 4-, and 5-dimensional Minkowski space [17]. They found dimensional reduction in all cases either to $d_{\mathrm{MM}} = 0$ if single-element subsets are assigned zero volume, or $d_{\mathrm{MM}} \approx 2$ if such subsets are excluded. Given the prevalence of dimensional reduction to a value of 2, the latter results seem compelling. However, in a recent analysis by David Meyer [18], he argues that prior calculations using the Myrheim-Meyer dimension that were indicative of dimensional reduction to a value of 2, such as those performed in [17], by himself [3], and one of us [19] contain certain flaws. Addressing those flaws, Meyer finds no dimensional reduction for causal sets generated by sprinklings in 3- and 4-dimensional Minkowski space.

尽管谱维度已经成为研究维度约化的首选方法，探究其他维度估计量的结果仍然很有意义。阿巴江和卡利普使用迈尔海姆-迈耶维度研究了可忠实嵌入 3、4、5 维闵氏空间的因果集小子集 [17]。他们发现在所有情况下都会发生维度约化: 如果单元素子集被赋予零体积，就约化为 $d_{\text{MM}} = 0$；如果排除这类子集，就约化为 $d_{\text{MM}} \approx 2$。鉴于维度约化到 2 是普遍结论，后者的结果看起来很有说服力。但在戴维·迈耶最近的分析 [18] 中，他指出此前使用迈尔海姆-迈耶维度、指向维度约化到 2 的计算 (比如文献 [17]、他自己的工作 [3] 以及我们一位作者的工作 [19]) 都存在一定缺陷。修正这些缺陷后，迈耶发现，对于撒播生成的嵌入 3 维和 4 维闵氏空间的因果集，不存在维度约化。

While there is some compelling evidence that causal sets may exhibit dimensional reduction to 2 on small scales, the issue is far from settled. Depending on the method for calculating the dimension and the way the results are analyzed, a clear picture has not yet emerged. As alluded to in [12], whether these kinematical dimension estimators can be found to give a consistent indication of small-scale dimensional reduction in causal sets, while it would be informative, is not the most important question. Ultimately, the effective dimension of spacetime on different scales in causal set theory will be dictated by a model of causal set dynamics. Perhaps lessons learned from these kinematical studies can help point the way to this dynamics.

虽然目前有一些有力证据表明因果集在小尺度下可能会发生维度约化到 2, 但这个问题远没有解决。根据维度计算方法和结果分析方式的不同，目前还没有形成清晰一致的结论。正如文献 [12] 暗示的，运动学维度估计量能否一致指示因果集的小尺度维度约化这件事，虽会提供不少信息，却并非最重要的问题。最终，因果集理论中不同尺度下的时空有效维度将由因果集动力学模型决定。或许这些运动学研究得到的经验能为找到这种动力学指明方向。

Asymptotic Silence

渐近沉默

The term asymptotic silence was given to the theoretical phenomenon that the lightcones of events near a singularity like that commonly featured in models of the early universe, effectively collapse (become more and more narrowly focused) as the singularity is approached [20]. As illustrated in Fig. 3, this collapsing of the lightcones causes worldlines that were once causally connected to become disconnected near the singularity. Therefore, communication between the events of these worldlines is cut off.

渐近沉默这一术语描述的是如下理论现象: 在早期宇宙模型普遍存在的奇点附近，随着不断趋近奇点，事件的光锥会发生有效坍缩 (变得越来越狭窄集中)[20]。如图 3 所示，光锥的坍缩会导致原本存在因果联系的世界线在奇点附近变得断开。因此，这些世界线上的事件之间的通讯会被切断。

A similar effect, short distance asymptotic silence, can occur on small spacetime scales, but for different reasons than in the early universe context [21, 22]. Such an effect would impact the effective dimension of spacetime at those scales. Interestingly, Carlip has argued that short-distance asymptotic silence could be the explanation for both dimensional reduction in many cases [15] and the increase in the causal spectral dimension of causal sets mentioned in section ”Spectral Dimension of Causal Sets” [23]. In the latter case, the onset of asymptotic silence would decrease the likelihood of the two random walkers to meet, thereby increasing the causal spectral dimension as determined by Eq. 33. Subsequently, Eichhorn et al. demonstrated that manifold-like causal sets do in fact exhibit behavior consistent with short-distance asymptotic silence [24].

短距离渐近沉默是一种类似效应，它可以出现在小时空尺度上，但产生原因和早期宇宙背景下的情况不同 [21, 22]。这种效应会影响这些尺度下时空的有效维度。有意思的是，卡利普曾提出，短距离渐近沉默既可以解释许多情况下的维数约化 [15]，也可以解释“因果集的谱维度”一节提到的因果集因果谱维度的升高 [23]。在后一种情况中，渐近沉默出现会降低两个随机游走者相遇的概率，从而提高由式 (33) 确定的因果谱维度。随后，艾希霍恩等人证明，类流形因果集确实表现出符合短距离渐近沉默的行为 [24]。

One of the lessons of this discussion is that even under circumstances not expected to be manifold-like, the results of different dimension estimators can convey important physical information. Whether particular causal sets show dimensional reduction and asymptotic silence are two examples. However, as causal set theory moves closer to a quantum dynamics, more such examples are likely to emerge.

本次讨论得到的结论之一是，即使在预期不属于类流形的环境中，不同维度估计量得到的结果也能传递重要的物理信息。特定因果集是否表现出维数约化和渐近沉默就是两个例子。随着因果集理论逐步趋近量子动力学，未来很可能会出现更多这类例子。

Some Performance Tests of Order-Theoretic Estimators

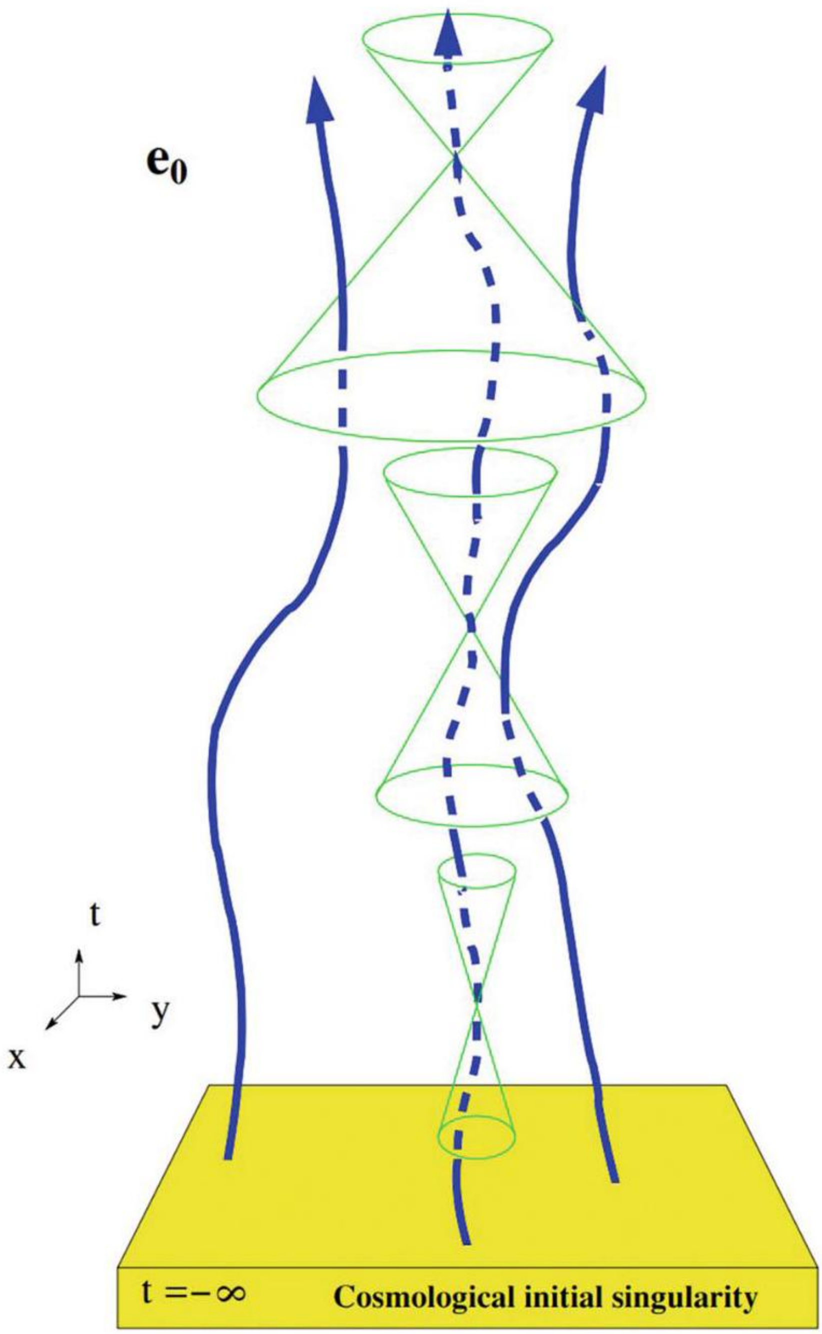
序理论估计量的若干性能测试

Here we discuss some quantitative results to convey how well the order-theoretic dimension estimators perform. To this end, Aghili et al. conducted simulations to test the precision and accuracy of four of the estimators discussed in this chapter: the constant cardinality approach (section ”Modified Myrheim-Meyer: Constant Cardinality Approach”), the midpoint scaling approach (section ”Midpoint Scaling Method”), the Brightwell-Gregory approach (section ”Brightwell-Gregory Method”), and their average path length approach (section ”Average Path Length Method”) [4]. These were compared for causal sets that ranged in size N from 100 to 1500 elements, and for manifold dimensions d ranging from 2 to 5 . All approaches improved with increasing N . Their results are shown in Fig. 4.

此处我们讨论一些定量结果，以说明序理论维数估计量的性能表现。为此，阿吉利等人开展了模拟测试，检验本章所讨论的四个估计量的精度与准确度: 定基数法 (章节“改进迈尔海姆-迈耶法: 定基数方法”)、中点标度法 (章节“中点标度法”)、布赖特韦尔-格雷戈里法 (章节“布赖特韦尔-格雷戈里法”)，以及平均路径长度法 (章节“平均路径长度法”)[4]。我们对这些方法在规模 N 从 100 到 1500 个元素、流形维数 d 从 2 到 5 的因果集上进行了对比测试。所有方法的性能都随 N 增大而提升，结果如图 4 所示。

Fig. 3 A visual representation of asymptotic silence in cosmology [20]. Focusing on the central worldline, as you go backward in time toward the initial singularity, the lightcones on this worldline become narrowly focused. The other, seemingly nearby, worldlines (along which this focusing also occurs) that were causally connected to the central worldline at later times become disconnected near the singularity. (C) IOP Publishing. Reproduced with permission. All rights reserved)

图 3 宇宙学中渐近沉默的可视化展示 [20]。聚焦中心世界线: 当你沿时间回溯至初始奇点时，该世界线上的光锥会逐渐收窄。其他看似邻近、同样会发生收窄的世界线，在较晚时刻还与中心世界线存在因果连接，在奇点附近会变得 disconnected。(C) IOP Publishing。经许可可复制。保留所有权利)



In their study, the constant cardinality approach (upper left) performed best, reliably estimating the correct dimension for all cases, and with very low scatter, even at low N , compared to the other estimators tested. The midpoint approach (upper right) reliably estimated the correct dimension for dimensions 2-4, but systematically overestimated the dimension in the $d = 5$ case. The Brightwell-Gregory approach (lower left) did fairly well in dimensions 2-4, but with greater scatter than the constant cardinality approach. In the $d = 5$ case, this method systematically overestimated the dimension, although the effect lessened with increasing N

. The average path length method (lower right) fared similarly to the Brightwell-Gregory approach, although it did somewhat better for the $d = 4$ and $d = 5$ cases, especially at low N . At the lowest $N, N = 100$ and $N = 200$, the dimension was underestimated by an amount comparable to the standard deviation in the $d = 4$ and $d = 5$ cases, which is unique among the estimators that were compared in [4].

在他们的研究中，定基数法 (左上) 性能最佳: 对比其他测试的估计量，即使在低 N 下，也能在所有场景中可靠估计出正确维数，且离散度极低。中点法 (右上) 能对 2-4 维可靠估计出正确维数，但在 $d = 5$ 场景中会系统性高估维数。布赖特韦尔-格雷戈里法 (左下) 在 2-4 维表现尚可，但离散度高于定基数法。在 $d = 5$ 场景中，该方法会系统性高估维数，不过这一效应会随 N 增大而减弱。平均路径长度法 (右下) 表现与布赖特韦尔-格雷戈里法相近，但在 $d = 4$ 和 $d = 5$ 场景中表现稍好，尤其是低 N 时。在最低 $N, N = 100$ 和 $N = 200$ 条件下，该方法会低估维数，低估幅度与 $d = 4$ 和 $d = 5$ 场景中的标准差相当，这是文献 [4] 对比的所有估计量中独有的特征。

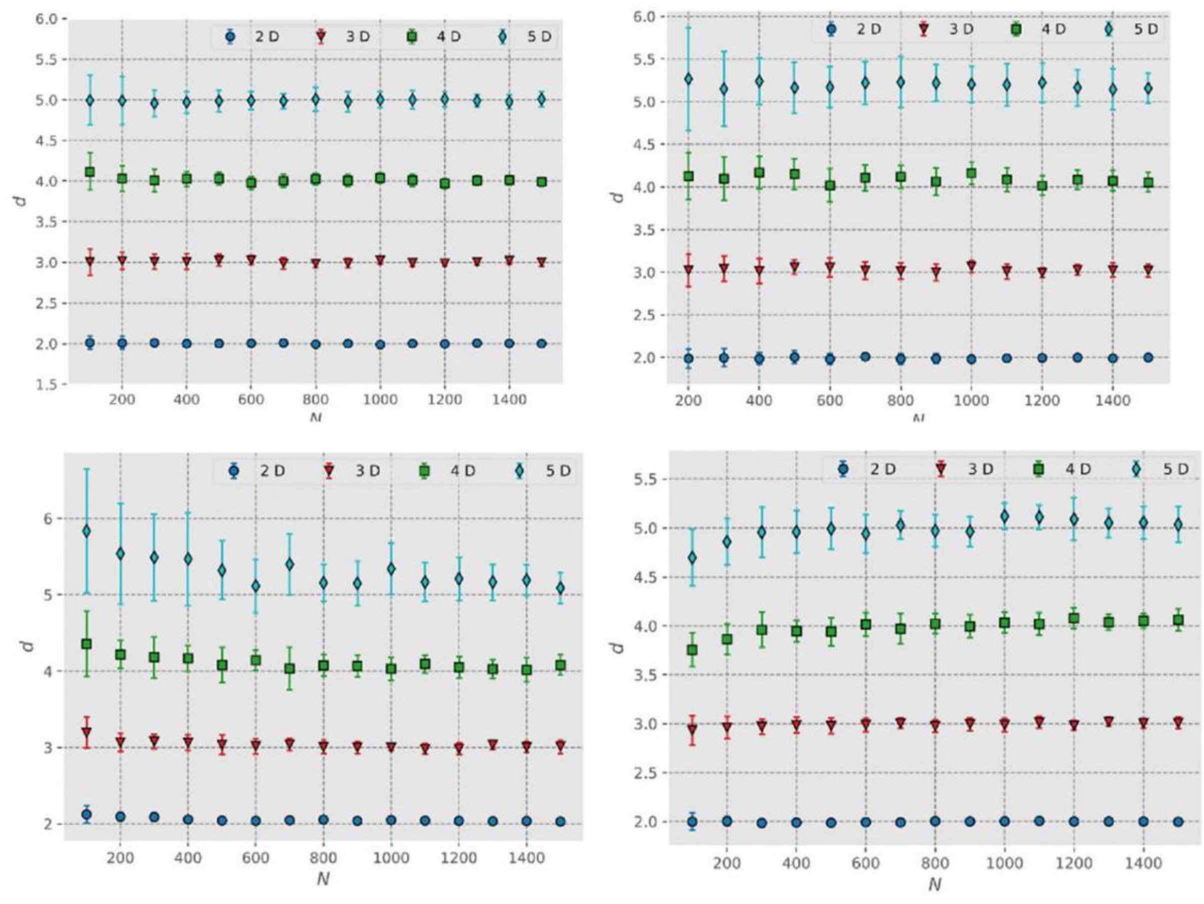


Fig. 4 Dimension vs. Cardinality for several manifold dimension estimators [4]. Plotted are the constant cardinality method (upper left), the midpoint scaling method (upper right), the Brightwell-Gregory method (lower left), and the average path length method (lower right). The error bars indicate the standard deviation of the results from 20 sprinklings. (C) IOP Publishing. Reproduced with permission. All rights reserved)

图 4 多种流形维数估计量的维数-基数关系图 [4]。图中依次为定基数法 (左上)、中点标度法 (右上)、布赖特韦尔-格雷戈里法 (左下)、平均路径长度法 (右下)。误差棒表示 20 次撒播结果的标准差。(C) IOP Publishing。经许可复制。保留所有权利)

For the remaining order-theoretic estimators, in Table 1 we provide numerical results of the Myrheim-Meyer dimension (section ”Myrheim-Meyer Dimension”) and the curved spacetime dimension estimator (section ”Modified Myrheim-Meyer: Curved Spacetimes”). We take the Myrheim-Meyer results from Reid [19], who compared the results of the Myrheim-Meyer and midpoint scaling dimensions for manifold-like causal sets embeddable in flat and conformally flat spacetimes as a function of subinterval size. That study found that the behavior of the estimators for small subintervals of causal sets formed from curved spacetimes matched well with the analogous behavior for causal sets formed from flat spacetimes of the same dimension. This way of estimating the dimension for causal sets embeddable in curved spacetimes is similar in principle to the technique using k -chains represented in Fig. 1.

对于其余序理论估计量，我们在表 1 中给出了迈尔海姆-迈耶维数 (章节“迈尔海姆-迈耶维数”) 和弯曲时空维数估计量 (章节“改进迈尔海姆-迈耶法: 弯曲时空”) 的数值结果。我们采用了里德研究 [19] 中迈尔海姆-迈耶法的结果，该研究对比了迈尔海姆-迈耶法与中点标度法，针对可嵌入平直和共形平直时空的类流形因果集，得到了维数随子区间大小变化的结果。该研究发现: 对于弯曲时空生成因果集的小子区间，估计量的行为与同维平直时空生成因果集的对应行为匹配良好。这种估计可嵌入弯曲时空因果集维数的方法，原理上与图 1 中用 k 链的技术类似。

Table 1 Numerical results for a few manifold dimension estimators. The Myrheim-Meyer and midpoint results from [19] were calculated from order intervals containing 512 elements. The curved results from [7] are from the generalized model with $\mu = 4$ and $k_1 = 1$

表 1 几种流形维数估计量的数值结果。文献 [19] 中的迈尔海姆-迈耶和中点结果，由包含 512 个元素的序区间计算得到。文献 [7] 的弯曲时空结果来自包含 $\mu = 4$ 和 $k_1 = 1$ 的广义模型

Dimension	2	3	4
Myrheim-Meyer	1.996	3.017	3.964
Midpoint	2.002	3.075	4.167
Curved	1.998 (dS_2)	3.005 (FLRW3)	

The curved spacetime results in Table 1 are from Kambor and X, who used a system of k -chains, see Eq. 19, to calculate the dimension of causal sets sprinkled in two-dimensional de Sitter (dS_2) with 25,600 elements and three-dimensional FLRW (FLRW3) with 102,400 elements. They tested the method with parameter (μ) values of 0.25,1, and 4 and starting k (k_1) values of 1,2,3, and 4 . They found that the correct dimensionality was obtained in all cases. The table shows the case with $\mu = 4$ and $k_1 = 1$, which corresponds to the original approach of [6] in Eq. 17.

表 1 中的弯曲时空结果来自 Kambor 与 X，他们使用 k 链系统 (参见式 19) 计算了撒播在二维德西特 (dS_2) 空间中含 25600 个元素、以及三维 FLRW(FLRW3) 中含 102400 个元素的因果集合的维度。他们对参数 (μ) 分别取 0.25、1、4，起始 k (k_1) 分别取 1、2、3、4 测试了该方法，发现在所有测试情形中都得到了正确的维度值。该表展示的是 $\mu = 4$ 和 $k_1 = 1$ 对应的情形，这对应式 17 中文献 [6] 的原始方法。

Concluding Remarks

总结

In this chapter, we have shown that there are now several effective methods for estimating the manifold dimension of causal sets. While many are designed to work for causal sets that approximate flat spacetimes, some promising ideas applicable to causal sets that approximate curved spacetimes have been proposed and tested [6, 7, 19] . We have also noted that various forms of spectral dimension might be well suited to the study of dimension-related phenomena on all scales, including the quantum regime. Moving forward, more quantitative investigations of the effectiveness of dimension estimation of curved spacetimes are needed. Beyond seeking the manifold dimension, research on the effective dimension of spacetime on intermediate and small scales in causal set theory will continue to be an active area of research.

在本章中，我们介绍了目前已存在多种估计因果集流形维数的有效方法。其中多数方法是为近似平直时空的因果集设计的，但也已有学者提出并测试了一些可应用于近似弯曲时空的因果集的可行思路 [6, 7, 19] 。我们也指出，多种形式的谱维数或许非常适合研究包括量子区域在内所有尺度上与维数相关的现象。未来仍需对弯曲时空维数估计的有效性开展更多定量研究。除寻找流形维数外，因果集理论中中等与小尺度下时空有效维数的研究仍将是一个活跃的研究领域。

Cross-References

交叉引用

The Philosophy of Causal Set Theory

因果集理论的哲学基础

Toward the Emergence of Continuum Spacetime in Causal Set Theory

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